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Block persistence

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Abstract. We define a block persistence probability $p_l(t)$ as the probability that the order parameter integrated on a block of linear size l has never changed sign since the initial time in a phase-ordering process at finite temperature $T < T_c$. We argue that $p_l(t) \sim l^{-z\theta_0} f(t/l^z)$ in the scaling limit of large blocks, where z is the growth exponent ($L(t) \sim t^{1/z}$), θ_0 is the global (magnetization) persistence exponent and $f(x)$ decays with the local (single spin) exponent θ for large x. This scaling is demonstrated at zero temperature for the diffusion equation and the large-n model, and generically it can be used to determine easily θ_0 from simulations of coarsening models. We also argue that θ_0 and the scaling function do not depend on temperature, leading to a definition of θ at finite temperature, whereas the local persistence probability decays exponentially due to thermal fluctuations. These ideas are applied to the study of persistence for conserved models. We illustrate our discussions by extensive numerical results. We also comment on the relation between this method and an alternative definition of θ at finite temperature recently introduced by Derrida [Phys. Rev. E **55**, 3705 (1997)].

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1 Introduction

Phase-ordering processes [1] correspond to the dynamics of systems quenched from a disordered high temperature state to a temperature where the equilibrium state is ordered. Dynamics proceed through coarsening of ordered domains and the domain linear scale $L(t)$ diverges as $t^{1/z}$. In the coarsening regime, nontrivial spatial and temporal correlations develop, adopting a scaling form. For instance, the order parameter equal time correlation function $\langle \varphi(\mathbf{x},t)\varphi(\mathbf{0},t)\rangle = f(|\mathbf{x}|/L(t))$.

Universality classes depend not only on the space dimension and the symmetries of the order parameter, as for static critical phenomena, but also on the conservation laws of the dynamics. Indeed, for a single static universality class, several dynamics can be used with the only constraint that they must obey detailed balance. For a scalar order parameter, nonconserved dynamics (model A) describe a ferromagnet, while conserved dynamics (model B) describe demixtion or segregation in binary alloys. Consequently, the set of dynamical critical exponents such as z and λ, defined by $\langle \varphi(\mathbf{x},t')\varphi(\mathbf{x},t)\rangle \sim [L(t)/L(t')]^{\lambda}$, for $t' \gg t$, is not related to static exponents by any hyperscaling law.

A remarkable point is that, as far as the temperature T of the quench is concerned, there are only two universality classes, namely $T = T_c$ (critical quenches) or $T < T_c$. This is assessed by numerics, renormalization group results or large-n expansions [1]. More precisely, two-point correlations have the same scaling (up to multiplicative constants) for any $T < T_c$. Therefore the temperature is an irrelevant parameter for $T < T_c$ quenches, and one may set $T = 0$ as well.

However, the situation is not as simple if one considers quantities involving more subtle correlations. One such quantity, which has attracted much interest recently, is the *persistence probability*, defined as the probability that the local order parameter at a given point $\mathbf{x}$ has never changed sign since the initial time [2–4]. For instance, in simulations of the Glauber Ising model, $p(t)$ is the fraction of spins that have never flipped since the initial time. At $T = 0$, $p(t)$ is the probability that a given point has never been crossed by a domain wall. It usually decays with an exponent $p(t) \propto t^{-\theta}$. For general nonequilibrium dynamics, $p(t)$ is the probability that a zero-mean stochastic quantity has never changed sign since the initial time. The analytical study of $p(t)$ is difficult due to the fact that it probes the whole history of the process. Even for simple scalar diffusion with a zero-mean random initial condition, a nontrivial algebraic decay is found [5,6]. At T_c, the persistence of the global magnetization was shown to yield a new independent critical exponent for the Ising model [7,8].

The temperature universality of the $T < T_c$ scaling of correlations, corresponding to a single fixed point $T = 0$

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in the renormalization group, seems to be broken for the local persistence $p(t)$, because at $T > 0$ thermal fluctuations lead to an exponential decay of p, in contrast to the power law decay at $T = 0$. To address this question, Derrida [9] recently proposed to study persistence at finite temperature for nonconserved Ising and Potts models by comparing two systems A and B evolving with the same thermal noise from two different initial conditions. System A is initially in a completely random configuration whereas B is in its fundamental state (all spins assuming the same value). Persistence is now defined as the probability $r(t)$ that $S_i^A S_i^B(t)$ has kept a constant sign since $t = 0$. The underlying idea is to discard simultaneous flips (at the same site) in both systems, because flips in B are only due to thermal fluctuations.

The implementation of this simple idea by Derrida [9], and more extensive simulations performed by Stauffer [10], have shown that $r(t)$ decays algebraically. For the Ising model, the observed exponent seems to be temperature-independent and equal to the $T = 0$ local persistence exponent θ. Therefore universality seems to hold with this new definition of persistence. However, the method cannot be used for *conserved models*, as system B would not evolve. Since Kawasaki (spin-exchange) dynamics freeze at zero temperature, a definition of persistence at finite temperature applying to conserved models is required. Moreover, Derrida's definition is not easy to generalize to continuous models. A definition involving a single system would be more satisfactory, as we know from the study of *damage spreading* that behaviors of observables obtained by comparison of two systems evolving with the same noise often depend on the Monte-Carlo algorithm used (see below).

In a recent letter [11], we introduced the notion of block persistence as a very natural method to give a temperature-independent and intrinsic definition of the persistence exponent. The method is in a way an *à la Kadanoff* implementation of the renormalization group ideas underlying the universality of correlations. The block persistence probability $p_l(t)$ is the standard persistence probability for a coarse-grained variable obtained by integrating the order parameter on a block of linear size l. In [11], we argued that the large l scaling of $p_l(t)$ is independent of T and corresponds to the $T = 0$ fixed point, because increasing l reduces the relative thermal fluctuations of the block variables.

In this article, we give a more detailed and general discussion of block persistence, which we illustrate with extensive simulations of different coarsening models. The structure of the paper is the following. We start by reviewing in Section 2 a few mathematical results needed to discuss persistence for physical models.

In Section 3, we comment further on Derrida's comparison method and check its intrinsicality. We show that even if the persistence exponent does not seem to depend on the algorithm used, the cross-over to T_c and the $T > T_c$ behavior of the persistence probability is completely different for heat-bath and Glauber dynamics.

In Section 4, we start from a general discussion of global persistence below T_c, and define block persistence as a natural way to include in a single framework the global and the local persistence exponent, through its scaling for $l \to \infty$ with $l/L(t)$ fixed at $T = 0$ and $T > 0$. At $T = 0$, we explicitly prove the postulated scaling form for the diffusion equation and the large-n model. We show that block scaling leads to an easy numerical determination of the global persistence exponent θ_0. We present numerical results for several systems, illustrating the previous discussion.

In Section 5, we move to finite temperature and justify that the scaling should be the same as at zero temperature, because the thermal exponential decay is eliminated in the scaling limit of large blocks. Thus block persistence provides a definition of local persistence at finite temperature. We present simulations for the Ising and Potts models, illustrating temperature universality. We also discuss the $T = T_c$ case.

In Section 6, we discuss the special case of conserved order parameter dynamics. Block scaling works as for the nonconserved case, but for an important feature: in the scaling, the global persistence exponent, which is zero for consrved models, must be replaced by a generally speaking nontrivial exponent θ'. This point is illustrated by simulations of one-dimensional models. We also show numerical results for two-dimensional Kawasaki dynamics at finite temperature.

2 Mathematical and general results

Before moving to physical problems, we would like to summarize a few useful mathematical results. Consider a general stochastic process $X(t)$, with $\langle X(t)\rangle = 0$. We are interested in the probability $p(t)$ that $X(t') > 0$ for all $0 \le t < t'$.

This is an old problem in probability theory [12,13], but a difficult one, and despite the large number of papers devoted to this subject, very few quantitative results are known, most of them concerning *stationary* and *Gaussian* processes, which are completely determined by their correlator $C(\tau) = \langle X(t)X(t+\tau)\rangle$. With these strong restrictions, $p(t)$ still cannot be computed analytically, even in the large t limit. Actually, $p[t, C(\tau)]$ is known only for very few specific correlators [12,13]. One of these correlators is $C(\tau) = e^{-a\tau}$, which is the general correlator of a *Markovian* stationary Gaussian process with the condition $C(0) = 1$, for which

$$p(t) = \frac{2}{\pi} \arcsin(e^{-at}) \tag{1}$$

and $p(t) \sim (2/\pi)e^{-at}$ at large t.

Generally speaking, $p(t)$ and its asymptotic large t decay depend sensitively on the *whole function* $C(\tau)$ and not only on its behavior for small or large τ. For instance, Majumdar and Sire [14] have considered the Gaussian process with $C(\tau) = (1-\varepsilon)e^{-\tau} + \epsilon e^{-2\tau}$. Despite the fact that C decays as $e^{-\tau}$ at large τ for all $\varepsilon < 1$, simulation of the

process shows that $p(t) \propto \exp(-a(\varepsilon)t)$, where $a(\varepsilon)$ interpolates continuously from 1 to 2 when ε is varied from 0 to 1. In [14], the Markovian correlator was used as a starting point for perturbative and variational approximations, which are however uncontrolled.

The following rigorous results for any stationary Gaussian process with zero mean are also very useful [12,13]:

$$p[t, bC(\tau)] = p[t, C(\tau)] \quad (2)$$

$$p[t, C(b\tau)] = p[bt, C(\tau)] \quad (3)$$

$$(\forall \tau, C_1(\tau) \geq C_2(\tau)) \Rightarrow (\forall t, p[t, C_1(\tau)] \geq p[t, C_2(\tau)]). \quad (4)$$

From the first relation, we see that p is completely determined by the normalized correlator $C(\tau)/C(0)$. The second relation will be used to obtain scaling forms for persistence probabilities in the following. Finally, the third relation shows that $p(t)$ decays exponentially in time for a stationary Gaussian process with a correlator that is bracketed for all τ by two Markovian correlators $e^{-b|\tau|} \leq C(\tau) \leq e^{-a|\tau|}$, because then $p(t)$ is also bracketed by two exponentials. Most of the correlators encountered in physical nonequilibrium processes actually have this property in a proper time variable (see below). However, there might be power law prefactors in the large t decay of $p(t)$.

Of course, in nonequilibrium dynamics, stochastic processes are scarcely Gaussian and, by definition, never stationary in physical time. However, if there is scaling relatively to a dynamically diverging scale $L(t)$, one must have for large t *and* t'

$$a(t,t') = \frac{\langle X(t)\rangle X(t')\rangle}{\sqrt{\langle X^2(t)\rangle \langle X^2(t')\rangle}} = f[L(t)/L(t')], \quad (5)$$

with $f(x) = f(1/x)$. This implies the stationarity of the process $X(t)/\sqrt{\langle X^2(t)\rangle}$ in the variable $u = \ln L(t)$.

Now if the process is Gaussian, we obtain that generically $p(u)$ decays as $e^{-\bar{\theta}u}$ and therefore $p(t)$ decays as $L(t)^{-\bar{\theta}}$. For most systems $L(t) \propto t^{1/z}$ and we recover the power law decay in time with $\bar{\theta} = z\theta$. The simplest example of such a Gaussian process is the diffusion equation (see below). Still, because the process is non Markovian, $\bar{\theta}$ cannot be computed analytically and an independent interval approximation was used to predict accurately $\bar{\theta}$ [5,6].

To end with this general discussion, we consider the following situation, which will be of use in the study of block persistence. Consider a family of Gaussian processes indexed by a variable $l > 0$, $\{X_l(t)\}$, with normalized correlators $a_l(t,t')$, with the following scaling property

$$a_l(t,t') = h(t/l^z, t'/l^z). \quad (6)$$

Then obviously $X_l(t) = X_1(t/l^z)$, leading to $p(t) = p_1(t/l^z)$.

3 Comparison of systems

Now let us come back to coarsening processes. Consider the nonconserved Ising dynamics. At $T = 0$, $p(t) \propto t^{-\theta}$, where θ is nontrivial and seems to be independent of other exponents. This is due to the fact that spins cannot flip when they are within an ordered domain. Flips occur only at interfaces between domains, and the slow surface tension driven motion of these interfaces makes for the slow decay of p.

The situation is dramatically different at finite temperature, because thermal fluctuations allow energetically forbidden flips. These activated flips occur with a decay rate $\tau \sim e^{-\Delta E/k_B T}$, where ΔE is a typical energy barrier to flip a spin inside a domain, of order the exchange constant J. Therefore, these thermal flips lead to an exponentially decaying $p(t) \propto e^{-t/\tau}$ (see Sect. 5).

Why then is there a unique scaling of correlations at finite $T < T_c$? The reason is that the domain structure in the scaling regime is the same at any $T < T_c$. The thermal fluctuations cancel out in the two point correlations, which reflect only the alternation of domains of different phases. The temperature dependence of the value of the bulk magnetization (approximately equal to its equilibrium value) just leads to a temperature dependent multiplicative constant in the scaling function.

From this point, it becomes clear that a simple temperature-independent definition of θ should be through the probability $r(t)$ that *a given site has never changed phase, i.e.* has never been crossed by a domain wall. At $T = 0$, we clearly have $r(t) = p(t)$, and at $T < T_c$ because of the universality of the domain dynamics, $r(t)$ should have the same decay as at $T = 0$.

Derrida [9] proposed a very clever scheme to implement this idea for the nonconserved Ising model by simulating two systems A and B evolving with the same Monte-Carlo dynamics, with *the same thermal noise*. System A is prepared in a completely random initial condition, whereas B is prepared in the fundamental state (all spins equal to one). Then both systems are updated simultaneously using the heat-bath algorithm with the same random number z at the same site i:

$$S_i^A(t + \Delta t) = \text{sign}\left[\frac{1 + \tanh(\beta \sum_i S_i^A(t))}{2} - z\right] \quad (7)$$

$$S_i^B(t + \Delta t) = \text{sign}\left[\frac{1 + \tanh(\beta \sum_i S_i^B(t))}{2} - z\right]. \quad (8)$$

Then, the fraction of persistent spins $r(t)$ is defined as the fraction of sites for which $S_i^A S_i^B$ has kept a constant sign since $t = 0$. It means that we discard flips that occur simultaneously in both systems, because flips in system B are purely thermal fluctuations, as there is a single + phase. Accordingly, Derrida found that at finite temperature $T < T_c$, $r(t) \propto t^{-\theta}$, with θ consistent with the $T = 0$ persistence exponent. This was confirmed by extensive simulations performed by Stauffer [10].

However, this practical definition of persistence for the Ising model is not completely satisfactory. First, it cannot be directly adapted to continuous models. Indeed, for a continuous order parameter the probability of a simultaneous flip in both systems will be zero in continuous time. A further restriction is that the method cannot be used for conserved dynamics, as the Kawasaki spin-exchange

dynamics, because system B would not evolve from a uniform initial condition. There is no proper initial condition for system B. This restriction is important, because Kawasaki dynamics cannot be studied at zero temperature, and therefore a definition of persistence at finite temperature is required. Finally, one would be more satisfied to get an intrinsic definition of persistence. The comparison of two systems evolving from different initial conditions has attracted much attention, especially in relation to the notion of damage spreading [15]. It was soon realized that the behavior observed depends on the implementation of the Monte-Carlo algorithm. Therefore, one could fear that Derrida's definition may work only with the heat bath algorithm. To check this, we performed simulations with the heat-bath algorithm and the Glauber algorithm. We find that the $T < T_c$ behavior is the same for both dynamics. However, quite interestingly, the $T \geq T_c$ behavior of $r(t)$ is completely different.

For heat-bath dynamics, for $T > T_c$, $r(t)$ reaches a plateau. This corresponds to the fact observed by Derrida and Weisbuch [16] that above T_c two systems evolving with this algorithm become identical within a finite time. When $T \to T_c$, this plateau crosses over to a power law $r(t) \sim t^{-\theta_c}$. From simulations at T_c we find $\theta_c \approx 0.9$, but θ_c can also be extracted from a scaling analysis of the cross-over for $T \to T_c^+$. At finite $T > T_c$, there are no domain walls. Starting from an infinite temperature state with a correlation length $\xi = 0$, ξ increases to reach its equilibrium value ξ_{eq}. In the vicinity of T_c, $\xi_{eq} \sim (T - T_c)^{-\nu}$ is very large. Therefore, at early times, for $\xi(t) \ll (T - T_c)^{-\nu}$ the system behaves as if it were to reach a critical (infinite ξ) equilibrium state, *i.e.* as if it were at T_c, and $\xi(t) \sim t^{1/z_c}$ while $r(t) \sim t^{-\theta_c}$. Deviations from this power law behavior appear only at late times when $\xi(t)$ approaches the finite value ξ_{eq} and $r(t)$ reaches a plateau. Consequently we expect the scaling form

$$r(t) \sim (\xi_{eq})^{\alpha} g[\xi(t)/\xi_{eq}] \sim t^{-\theta_c} f[t(T - Tc)^{\nu z_c}], \qquad (9)$$

where $f(x) \propto x^{\theta_c}$ when $x \to \infty$ and $f(x)$ tends to a constant when $x \to 0$.

This scaling behavior is illustrated in Figure 1, which shows results of simulations of heat bath dynamics for the two dimensional Ising model at different temperatures above T_c. The best scaling is obtained with $\theta_c = 0.9$ ($\nu = 1$ and $z_c = 2.17$). It is quite surprising to obtain a new exponent at T_c (see the discussion for block persistence below), and one should wonder whether this exponent is universal or if it depends on the chosen algorithm.

If we consider another frequently used algorithm, the Glauber dynamics, which corresponds to the update rule,

$$S_i(t + \Delta t) = S_i(t)\,\mathrm{sign}\left[\frac{1 + \tanh(\beta S_i \sum_j S_j(t))}{2} - z\right] \qquad (10)$$

and using the same z at the same site for system A and B, we find a very different behavior of $r(t)$ above T_c. As shown in Figure 2, it now decays faster than any power

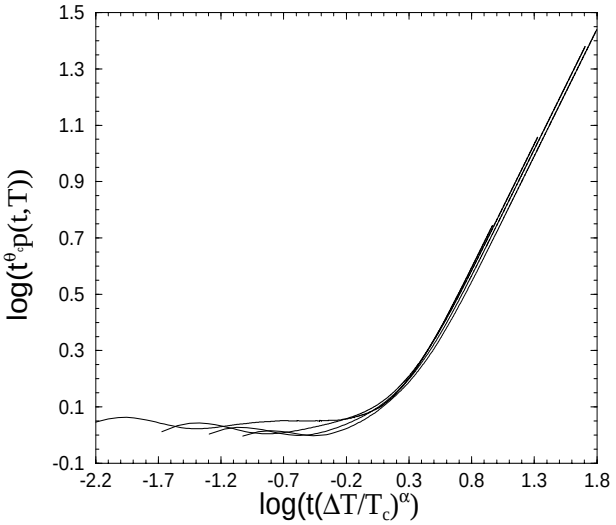

Fig. 1. Scaling behavior of the fraction of persistence spins $r(t)$, from Derrida's definition, for the Ising model with heat-bath dynamics when $T \to T_c^+$. Simulations were carried out on a 1000^2 lattice and 20 samples were averaged. The data collapse is obtained with $\theta_c = 0.9$, $\alpha = \nu z_c$, with the exact value $\nu = 1$, and $z_c = 2.17$. The scaling function goes to a constant at small argument and diverges as a power law at large argument.

law. At T_c, it is difficult to distinguish from our simulations performed on a 1500^2 lattice, averaging over 20 samples, whether $r(t)$ decays exponentially or as a power law with an exponent θ_c bigger than 2.4 (the value extrapolated from our data). Anyway, we do not find the exponent $\theta_c = 0.9$ found for heat-bath dynamics. Thus, this exponent is not intrinsic, neither is the $T > T_c$ behavior of $r(t)$ (similar results have also been found by Hinrichsen and Antoni [17]).

This illustrates the kind of problems that can be encountered using observables defined by the comparison of two systems. Note however that, as said before, the large t decay of $r(t)$ below T_c is the same for both dynamics, and therefore seems to be intrinsic. The cross-over in the vicinity of T_c^- will be, of course, different. One could be tempted to relate the different behavior obtained above T_c to damage spreading properties of the dynamics. The question of damage spreading is to know whether the distance (in configuration space) of two copies of a same system, evolving from two slightly different initial conditions, diverges (damage is said to spread), or keeps bounded (damage is said to heal). For the two-dimensional Ising model, damage heals for heat-bath dynamics and spreads for Glauber dynamics. However, in one dimension, damage heals for both dynamics, and we have checked that both dynamics lead to an exponential decay of $r(t)$. Therefore, the absence of damage spreading is not a sufficient condition to obtain the saturation of $r(t)$, which seems to be due to a very specific property of heat-bath dynamics in two dimensions.

In the rest of the article, we describe a completely different approach to finite temperature persistence,

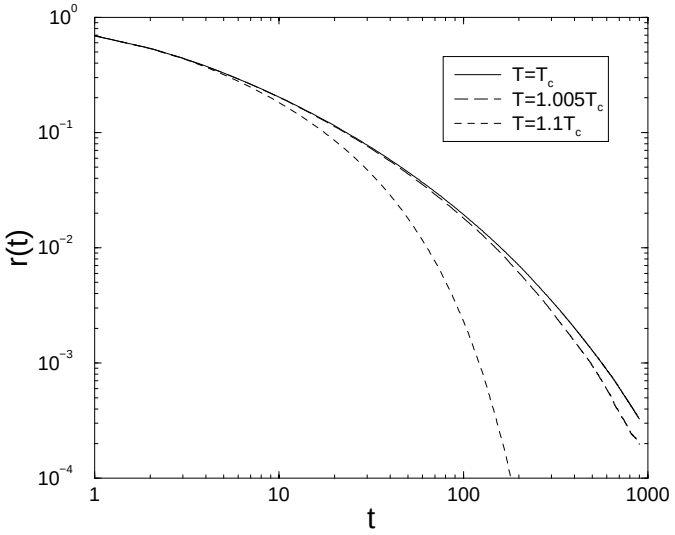

Fig. 2. Decay of the fraction of persistent spins $r(t)$, with Derrida's definition, for the Ising model with Glauber dynamics at $T = T_c$, $1.005T_c$ and $1.1T_c$, from simulations on a 1500^2 lattice (20 samples). Above T_c, $r(t)$ decays faster than any power law, in contrast with heat-bath dynamics. At T_c, $r(t)$ also seems to decay faster than algebraically, but one cannot positively rule out a power law decay with a large exponent θ_c bigger than 2.4 (the extrapolated exponent from our data).

involving a *single system*, which therefore avoids such difficulties and can be applied to continuous and conserved models.

4 Block persistence at zero temperature

We consider the nonequilibrium dynamics of a *nonconserved* order parameter $\varphi(\mathbf{x}, t)$, which can be either continuous or discrete. The case of a locally conserved order parameter will be postponed to Section 6. We first discuss block persistence at $T = 0$, because as we will see later the scaling properties of p_l are temperature independent.

Actually, introducing block persistence is also quite natural at zero temperature, because it provides a link between the global and the local persistence probabilities. Block scaling will be shown below to be a very effective way of determining the global persistence exponent θ_0, corresponding to the global order parameter $M(t) = \int \varphi(\mathbf{x}, t)d\mathbf{x}$. This global persistence exponent has been extensively studied at T_c [7,8,18,19]. Here, we would like to thoroughly discuss the $T < T_c$ case. We shall assume $T = 0$, but the discussion would be the same at any $T < T_c$.

4.1 Global persistence

First, we remark that while the stochastic process $\{\varphi(\mathbf{x}, t)\}$ (at a given point) is generally speaking both non Gaussian and non Markovian, $\{M(t)\}$ is always Gaussian in the thermodynamic limit. Indeed, the magnetization vector $\mathbf{M} = (M(t_1), ..., M(t_n))$ is the sum of an infinite number of random vectors $\Phi(\mathbf{x}) = (\varphi(\mathbf{x}, t_1), ..., \varphi(\mathbf{x}, t_n))$. Since the correlation length for φ is finite at t_1, ..., t_n, the Φ vectors have short range correlations and the central limit theorem entails that the magnetization vector $\mathbf{M}$ is Gaussian, for every choice of an arbitrary number n of times. This is precisely the *definition* of the fact that the whole *process* $\{M(t)\}$ is Gaussian (which is stronger than just saying that $M(t)$ is Gaussian at any t). For a finite system, there are non Gaussian corrections due to the fact that the number of independent contributions to the magnetization vector is finite and of order $V/L(t_n)^d$, if t_n is the largest of the t_i. These non Gaussian contributions will be important only at long times of order $V^{z/d}$.

Therefore, for an infinite system, the global persistence probability is completely determined by the two-time correlator

$$a_g(t, t') = \langle M(t)M(t')\rangle / \sqrt{\langle M^2(t)\rangle \langle M^2(t')\rangle}. \quad (11)$$

The analytical determination of θ_0 is consequently simpler in principle than for θ (for θ, a nonlinear Gaussian approximation was used by Majumdar and Sire [14]). At $T = T_c$, Majumdar *et al.* [7] have been able to compute an ε expansion of the the global exponent θ_c for model A. For $T < T_c$, there is no natural perturbation parameter as ε. The analytical study of θ_0 can be performed using the methods of [14] in dimensions $d \geq 3$ [20].

Interestingly there is a relation between the autocorrelation exponent λ and θ_0 *when* M *is a Markov process*

$$\theta_0 z = \lambda - d/2. \quad (12)$$

This relation is the consequence of the scaling of correlations and is the counterpart of a similar scaling law at T_c [7].

To show equation (12), we use the fact that the Gaussian process $M(t)/\sqrt{\langle M(t)^2\rangle}$ is stationary in the scaling limit as a function of $u = \ln L(t)$ (see Sect. 2), with

$$a_g(t, t') = f[L/L'] = c(|u - u'|). \quad (13)$$

The two point correlator $C(\mathbf{k}, t, t') = L^\lambda {L'}^{d-\lambda} g(kL')$ in the scaling regime for $t' \gg t$. If $g(0) = O(1)$, which is the case for nonconserved models, $\langle M(t)M(t')\rangle \propto [L/L']^\lambda {L'}^d$ while $\langle M^2(t)\rangle \propto L^d$, and we obtain,

$$a_g(t, t') \sim \left[\frac{L}{L'}\right]^{\lambda - d/2} \sim \left[\frac{t}{t'}\right]^{(\lambda - d/2)/z}, \text{ for } t' \gg t. \quad (14)$$

Up to this point, the results are general and valid for a non Markovian process. Now, if the normalized M is Markovian, then necessarily $c(|u - u'|) = \exp(-\theta_0 z|u - u'|)$ (see Sect. 2). In other words $f(x) = x^{-\theta_0 z}$ for all $x \geq 1$, and since equation (14) expresses that $f(x) \sim x^{(d/2-\lambda)}$ for $x \gg 1$, we obtain equation (12). Note that the lower bound $\lambda \geq d/2$ proposed by Fisher and Huse [21] ensures that θ_0 is nonnegative. Below, we shall demonstrate that M is Markovian for the $T = 0$ one dimensional Glauber model, and we shall find that $\theta_0 z = 1/2 = \lambda - d/2$ ($\lambda = 1$),

but, generally speaking, equation (12) is violated because $M(t)$ is non Markovian and $f(x)$ is not a pure power law. The Markovian value of θ_0 is neither an upper nor a lower bound (see numerical results in Sect. 4.5).

The direct determination of θ_0 is quite difficult. One has to record the time when the magnetization first changes sign, for a very large number of runs, which limits drastically the sample size. Cornell and Sire [22] simulated the two dimensional Ising model on a $L = 8$ to 128 lattice, and were obliged to use a finite-size scaling analysis that did not prove very conclusive, leading to a large uncertainty on the value of $\theta_0 \approx 0.06 \sim 0.11$. We shall see below that block persistence, which we now define, leads to a much easier determination of θ_0.

4.2 Block persistence

The idea is to define a more general quantity, the *block persistence probability*, that coincides with the global and the local persistence in different limits. The procedure is very natural: we consider a coarse-grained variable $\varphi_l(\mathbf{x}, t)$, obtained by integrating scales smaller than l. The simplest procedure is to integrate φ over *a block of linear size* l, as will be done for numerical simulations of lattice spin models. Alternatively, one can also eliminate Fourier modes of wavelength smaller than l, as will be more convenient for the analytical treatment of continuous models. The block persistence probability $p_l(t)$ is just the persistence probability for the coarse-grained variable. For $l = \infty$ we recover the global persistence, while for $l = 0$ (or 1 on a lattice) we get the local persistence.

Now, for finite l, the time dependence of p_l interpolates between the two exponents θ and θ_0. Indeed, at early times, when $L(t) \ll l$, the system effectively sees infinite blocks, and $p_l(t) \propto t^{-\theta_0}$. Then for $L(t) \gg l$, blocks behave as single spins, and $p_l(t) \sim c_l t^{-\theta}$. Therefore, we expect a scaling form of $p_l(t)$ for $l \to \infty$ with a fixed ratio $l/L(t)$

$$p_l(t) \sim l^{-\alpha} g(L(t)/l) = l^{-\alpha} f(t/l^z), \tag{15}$$

where $f(x) \propto x^{-\theta_0}$ when $x \to 0$ and $f(x) \propto x^{-\theta}$ when $x \to \infty$. α must be equal to $z\theta_0$ because for finite t and $l \to \infty$, $p_l(t)$ must tend to the global persistence probability.

This scaling form can be demonstrated for two analytically tractable models closely related, namely the diffusion equation and the large-n limit of the $O(n)$ nonconserved model. The reason is that in both models, the process $\{\varphi(\mathbf{x}, t)\}$ is Gaussian, entailing that all coarse-grained variables are also Gaussian, and $p_l(t)$ depends solely on the normalized correlator

$$a_l(t, t') = \frac{\langle \varphi_l(\mathbf{x}, t)\varphi_l(\mathbf{x}, t')\rangle}{\sqrt{\langle \varphi_l^2(\mathbf{x}, t)\rangle \langle \varphi_l^2(\mathbf{x}, t')\rangle}}, \tag{16}$$

which can be computed analytically.

4.3 Diffusion equation

The diffusion equation may be the simplest example of nonequilibrium dynamics. It is not really a coarsening model, because of the absence of domain walls due to the linearity of the equation. Consider a scalar field φ evolving according to

$$\frac{\partial \varphi}{\partial t} = \nabla^2 \varphi, \tag{17}$$

from a random initial condition with zero mean $\langle \varphi \rangle = 0$ and short range correlations $\langle \varphi(\mathbf{x}, 0)\varphi(\mathbf{x}', 0)\rangle = \Delta\delta(\mathbf{x} - \mathbf{x}')$. For this model, the global magnetization is conserved, leading to $\theta_0 = 0$.

Integrating the equation in Fourier space, we obtain the Fourier transform of the correlator

$$C(\mathbf{k}, t, t') = \langle \tilde{\varphi}(\mathbf{k}, t)\tilde{\varphi}(-\mathbf{k}, t)\rangle = \Delta e^{-k^2(t+t')}. \tag{18}$$

Computing the two time correlator

$$C(t, t') = \sum_{\mathbf{k}} C(\mathbf{k}, t, t')$$

leads to the normalized correlator

$$a(t, t') = \frac{C(t, t')}{\sqrt{C(t, t)C(t', t')}} = \left(\frac{4tt'}{(t + t')^2}\right)^{\frac{d}{4}}. \tag{19}$$

This correlator yields a nontrivial persistence exponent, which can be reproduced with an excellent precision using an independent interval approximation (IIA) [5,6].

As remarked before, considering blocks of size l is equivalent to introduce an upper cut-off in Fourier space $\lambda \sim 1/l$, and to consider

$$\varphi_l = (1/\sqrt{V}) \sum_{|\mathbf{k}|<\lambda} \tilde{\varphi}(\mathbf{k}) \exp(i\mathbf{k}.\mathbf{x}).$$

The correlator of the corresponding block variables is,

$$C_l(t, t') = \langle \varphi_l(\mathbf{x}, t)\varphi_l(\mathbf{x}, t')\rangle = \sum_{|\mathbf{k}|<\lambda} C(\mathbf{k}, t, t'). \tag{20}$$

The φ_l variables are Gaussian, and the behavior of $p_l(t)$ depends only on the normalized correlator $a_l(t, t')$.

$$a_\lambda(t, t') = \left(\frac{\sqrt{tt'}}{t + t'}\right)^{\frac{d}{2}} \frac{F(\lambda^2(t + t'))}{\sqrt{F(\lambda^2 t)F(\lambda^2 t')}} = H(\lambda^2 t, \lambda^2 t') \tag{21}$$

with $F(x) = \int_0^x y^{d-1} e^{-y^2} dy$. From the final discussion of Section 2, we have $p_l(t) = p_1(t/l^2)$, which is precisely the scaling form assumed from physical arguments, with $\alpha = 0$.

The probability $p_l(t)$ cannot be explicited, but we can obtain its asymptotic behavior. For $t, t' \gg l^2$, or for $\lambda \to \infty$ one has

$$a_\lambda(t, t') \sim a(t, t') \tag{22}$$

i.e., we recover the one point two-time normalized correlator, leading to a nontrivial exponent θ. Then, in the opposite limit of large blocks (or small times) $t, t' \ll l^2$, one has $a_\lambda(t, t') = 1 + O(\lambda^2(t + t'))$, which corresponds to a nonevolving field. The scaling of the block persistence probability is therefore conform to the general discussion above.

4.4 Large n limit

Now, let us investigate the $O(n)$ model in the large n limit. As usual, we start from a n-components vectorial order parameter $\boldsymbol{\varphi}$ with the Time Dependent Ginzburg-Landau dynamics,

$$\partial_t \varphi_\alpha = \nabla^2 \varphi_\alpha - r\varphi_\alpha - \frac{g}{n}\varphi_\alpha \boldsymbol{\varphi}^2. \tag{23}$$

In the large n limit, $\boldsymbol{\varphi}^2/n$ can be replaced by the average $\langle\varphi^2\rangle$, where φ is now any component of the field, and one obtains a linear self-consistent equation, which reads in Fourier space,

$$\partial_t \tilde{\varphi}(\mathbf{k},t) = -(k^2 + R(t))\tilde{\varphi}(\mathbf{k},t), \tag{24}$$

with $R(t) = r + g\langle\varphi^2\rangle$.

Hence, $\tilde{\varphi}(\mathbf{k},t) = \tilde{\varphi}(\mathbf{k},0)h(t)^{-1/2}\exp(-k^2 t)$ with $h(t) = \exp(2\int_0^t R(t')dt')$. The self-consistence condition,

$$\sum_{\mathbf{k}} |\tilde{\varphi}(\mathbf{k},t)|^2 = VS(t), \tag{25}$$

with the definition $\varphi(\mathbf{k}) = (1/\sqrt{V})\int \varphi(\mathbf{x})d^d\mathbf{x}$, leads to the deterministic differential equation for $h(t)$,

$$\frac{1}{2}\dot{h} = rh + \frac{g\Delta}{V}\sum_{\mathbf{k}} e^{-2k^2 t}. \tag{26}$$

The global magnetization $m(t) = \tilde{\varphi}(\mathbf{0},t)/\sqrt{V}$ is just given by $m(t) = m(0)/\sqrt{h(t)}$. Therefore, $m(t)$ is deterministic (apart from the randomness of $m(0)$), and never changes sign, which yields $\theta_0 = 0$. Equation (26) can be solved using Laplace transform, but we do not need to know $h(t)$ here. The two-time correlator is

$$C(\mathbf{k},t,t') = \langle\tilde{\varphi}(\mathbf{k},t)\tilde{\varphi}(-\mathbf{k},t)\rangle = \frac{\Delta e^{-k^2(t+t')}}{\sqrt{h(t)h(t')}}\cdot \tag{27}$$

The Gaussian process $\bar{\varphi} = \varphi/h(t)$, has the same correlator as the diffusion equation. More precisely, $\bar{\varphi}$ obeys the diffusion equation. Hence the rest of the demonstration is the same as above. The persistence exponents θ, $\theta_0 = 0$ and the scaling function f are the same as for the diffusion equation.

In these two soluble models, the scaling law of equation (15) is valid for any t and any l, and not only asymptotically in the large l large t limit as will be the case in general. Remark also that the $\theta_0 = 0$ result recovers two different behaviors of the global magnetization. For the diffusion equation, the magnetization is exactly conserved, whereas for the large n model it relaxes deterministically to zero.

4.5 Results for global persistence

Thanks to the scaling form of equation (15), it is possible to use block scaling to compute θ_0 numerically. One evaluates $p_l(t)$ for different l, which can be done on a single run, and then adjusts θ_0 to obtain the best data collapse. We present here some numerical results for three different models, illustrating the three possible cases: $\theta_0 z$ equal to, bigger than or smaller than $\lambda - d/2$. We also give a direct derivation of the exact result $\theta = 1/4$ for the one dimensional Glauber dynamics (Majumdar *et al.* [7] used an interface representation of the dynamics). Finally we show a surprising relation between θ_0 for the one-dimensional XY model with power-law initial spatial correlations and θ for the diffusion equation.

One-dimensional Glauber model

One-dimensional coarsening is quite special, since the critical temperature is zero. For the Glauber Ising model, which is exactly soluble, the exact computation of θ was really difficult, while θ_0 is trivial since the global magnetization $M(t)$ is Gaussian at any time, and Markovian in the scaling limit. To show it, we just have to write the evolution equation for the two-point correlation [23], for $t > t'$,

$$2\frac{\partial C}{\partial t}(r,t,t') = C(r+1,t,t') + C(r-1,t,t') - 2C(r,t,t'), \tag{28}$$

with $C(r,t,t') = \langle S_r(t')S_0(t)\rangle$. Summing over r, we get,

$$2\frac{\partial\langle M(t)M(t')\rangle}{\partial t} = 0, \tag{29}$$

hence, $\langle M(t)M(t')\rangle = \langle M^2(\min(t,t'))\rangle$. Then, in the scaling regime $\langle M^2(t)\rangle \propto t^{1/2}$, leading to

$$a_g(t,t') = \left(\frac{t'}{t}\right)^{1/4}, \text{ for } t > t. \tag{30}$$

The normalized correlator of the global magnetization is equal to $\exp[(u'-u)/4]$ in the variable $u = \ln t$. This proves that the Gaussian process $M(u)$ is stationary and Markovian and that $p(u) \sim \exp(-|u|/4)$. In the t variable we get $\theta_0 = 1/4 = (\lambda - d/2)/z$, since $\lambda = 1$ and $z = 2$. Remark that while $M(t)$ is Markovian, $S(t)$ at a given point is not, neither is it Gaussian, and the computation of θ was a real tour de force [24] which cannot be extended to other systems.

To check the scaling assumption of equation (15) with the exact value of θ_0, we have simulated the Glauber Ising model on a 200 000 spins chain with block size 1, 21, 41, 61, 91 (Fig. 3). Ten samples were averaged to obtain the final data. The data collapse with $\theta_0 = 1/4$ is very good and the scaling function has the expected behavior: a power law divergence with exponent θ_0 at small argument and an algebraic decay with exponent θ at large argument.

One-dimensional model A

In one dimension, deterministic and stochastic models are known to lead to different growth laws and correlations [25]. For instance, in the one dimensional noiseless model A, domain walls have a weak attractive interaction decreasing exponentially with the distance, and

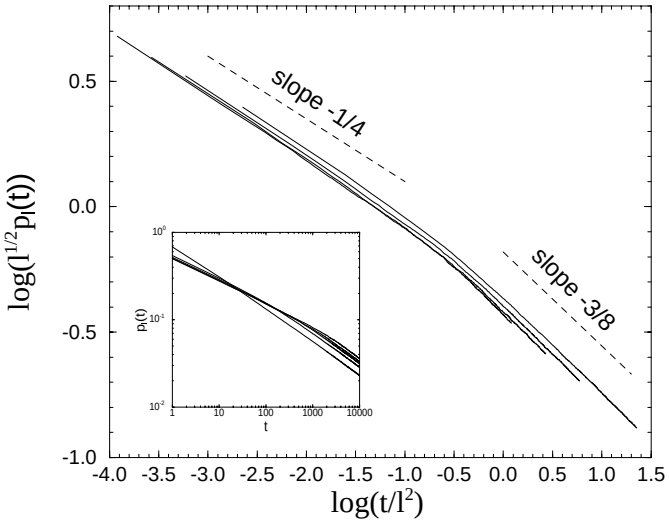

Fig. 3. Scaling of the block persistence probability $p_l(t)$ for a one-dimensional spin chain (200 000 spins, 10 samples), with block size $l = 1, 21, 41, 61, 91$ (from bottom to top in the right part of the insert). $l = 1$ is omitted in the scaling, and the data collapse improves as the block size increases. As expected $p_l(t) \sim t^{-\theta_0}$ for $t \ll l^2$ and $p_l(t) \sim t^{-\theta}$ for $t \gg l^2$ (with $\theta_0 = 1/4$ and $\theta = 3/8$).

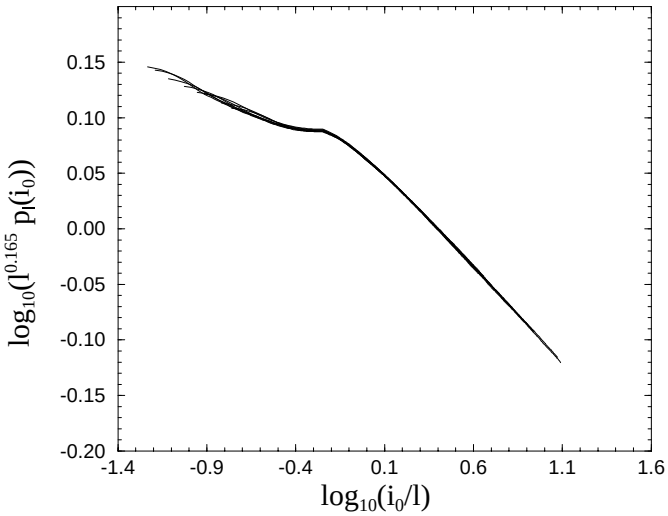

Fig. 4. Scaling of the block persistence probability obtained from simulations of the domain aggregation model equivalent to the deterministic one-dimensional model A, on a chain with 10^6 sites, $l = 71, 81, 101, 111, 131, 161, 191, 211$. Excellent scaling is found for $\theta_0 = 0.165$.

$L(t) \propto \ln t$, whereas in the discrete stochastic Glauber models, walls do not interact but perform simple random walks and annihilate when they collide, leading to $L(t) \propto \sqrt{t}$.

Model A is interesting, because it can be mapped on a simple deterministic model of charge aggregation [25–29]. In this model, domains of "+" and "−" phase evolve the following way. At each step, the smallest domain I_0 (length $l(I_0)$) is changed sign and merged with its two neighbors I_1 and I_2, to give a domain of length $l(I_0) + l(I_1) + l(I_2)$. To compute the domain size distribution, the sign of the domains can be forgotten and one can easily show that no correlations develop in the system. The mean-field equations are exact and can be solved for the scaling function of the size distribution. In this model, the time variable is the minimum length l_0.

Bray *et al.* have shown that the local persistence exponent θ [28] and the autocorrelation λ [29] have a geometrical interpretation in this model. For instance, defining for each domain the fraction of persistent spins $d(I)$, the new interval obtained in one step of the aggregation model has $d(I) = d(I_1) + d(I_2)$, and the total fraction of persistent spins can be computed in mean field since there are no correlations for d as well. The exact results are $\theta = 0.17504588...$ and $\lambda = 0.6006165...$ These exponents are solutions of implicit nonlinear equations.

As far as the global persistence exponent is concerned, we have to consider explicitly "+" and "-" domains. Spatial correlations are still irrelevant, and the sign-length distribution function evolves according to rate equations [29]. However, this simplification is not sufficient to allow the computation of θ_0. To understand the origin of the difficulty, let us consider a discrete lattice leading to integer values of $l(I)$. To increase the minimum domain length in the system from l_0 to $l_0 + 1$, one has to remove $n(l_0)$ domains. In each coalescence event, l_0 spins change sign leading to $\delta M = \pm\, 2l_0$, depending on the sign of the domain. The total change in magnetization ΔM is therefore equal to $-2l_0\, \delta\rho(l_0)$, where $\delta\rho(l) = n_+(l) - n_-(l)$ is the (algebraic) excess density of positive domains of length l. This quantity, as well as the magnetization itself $M = \sum l\, \delta\rho(l)$, is a *fluctuation* around a zero mean value, and is expected to be Gaussian (for an infinite system), with a variance $\langle \delta\rho^2 \rangle \propto \bar{n}(l)$, where $\bar{n}(l)$ is the total average density of domains of length l.

It is easily shown from the mean-field equations that $\bar{n}(l_0) \propto 1/l_0^2$ in the scaling (large l_0 regime). Therefore, the magnetization increment is a Gaussian variable with variance $2l_0\bar{n}(l_0)$ of order 1. If we assume (i) no long range correlations between the increments and (ii) no correlation between M and its increments, at least for small M, then M performs a simple random walk, leading to $\theta_0 = 1/2$. While assumption (i) is very reasonable, assumption (ii) is much more questionable, as it seems intuitive, although not compelling, that $\delta\rho(l_0)$ is more likely to have the sign of M, leading to a faster decay of the global persistance and $\theta < 1/2$. This intuition is confirmed by numerical simulations of the domain aggregation model. Block persistence data are presented in Figure 4 for a $L = 10^6$ chain (30 samples). The best scaling was obtained for $\theta_0 = 0.165$, much smaller than $1/2$. The profile of the scaling function in the cross-over region is quite different from the stochastic Glauber model. The Markovian scaling law would lead to $\theta_0 = \lambda - d/2 \approx 0.1$ ($z = 1$ since the time variable is the dynamical length scale). Hence, for this model we have $\theta_0 > \lambda - d/2$. To try to determine θ_0 analytically, one would have to compute the correlations of the fluctuations $\delta\rho$, which does not seem easy, even using

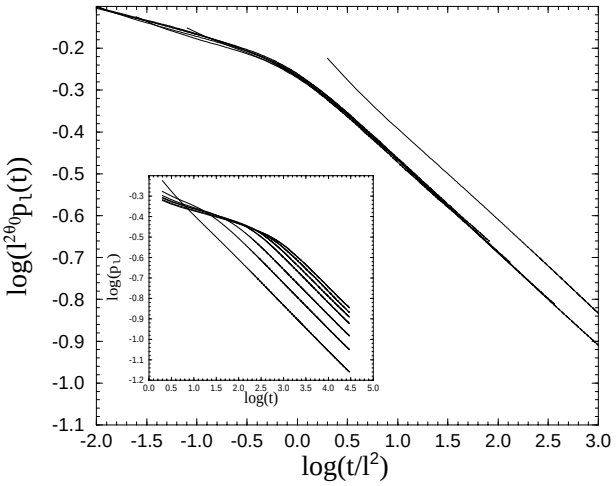

Fig. 5. Block persistence at $T = 0$ obtained from simulation of the nonconserved Ising model on a 2000^2 lattice, for l =1, 5, 9, 15, 19, 25, and 31 (from bottom to top in the insert). $p_l(t)$ decays as $t^{-\theta_0}$ at early time and as $t^{-\theta}$ at large time. Excellent scaling is then obtained taking $\theta_0 = 0.09$.

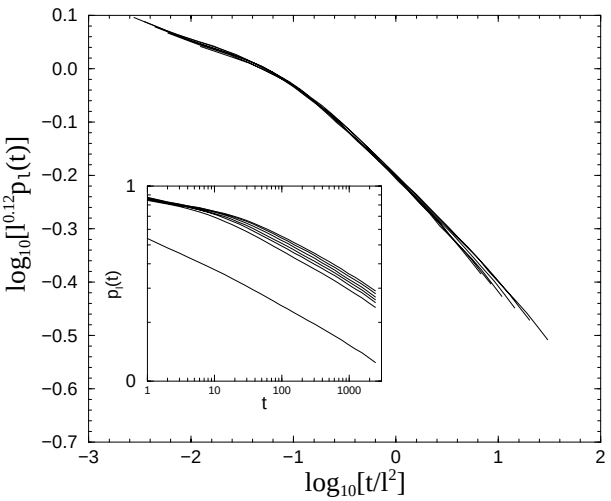

Fig. 6. Block persistence at $T = 0$ obtained from simulation of the time dependent Ginzburg-Landau equation at zero temperature on a 700^2 grid, for l =9, 11, 13, 15, 17 and 19 (from bottom to top in the insert). The scaling presented is obtained taking $\theta_0 = 0.06$. The large time decay of $p_l(t)$ corresponds to $\theta = 0.2$, with significant curvature though.

the mean-field rate equations as dynamics from an initial random domain distribution.

Two-dimensional Glauber Ising model

$d = 1$ is quite special because $T = 0$ is also the critical temperature, and persistence can only be defined at $T = 0$. Now we move to the $d = 2$ Glauber Ising model, for which block scaling will lead to a definition of θ at finite temperature. It is also interesting to determine θ_0 which could only be roughly evaluated by the direct method [22] despite much numerical effort. Comparatively, block scaling is a very easy and reliable method. We performed simulations on a 2000^2 lattice with blocks of linear size 1, 5, 9, 15, 19, 25, and 31. 20 samples were averaged to obtain the final data presented in Figure 5. We find excellent scaling, with $\theta_0 = 0.09$. The uncertainty in the data collapse is roughly of 1% on θ_0. This value of θ_0 is compatible with the range $0.06 \sim 0.11$ found by Cornell and Sire [22]. The Markovian value of $z\theta_0$ would be $11/8 = 1.375$ ($\lambda = 5/4$), and for this model we have $\theta_0 z < d - \lambda/2$.

Two-dimensional Ginzburg-Landau equation

We can also simulate the time-dependent Ginzburg-Landau equation (Fig. 6), corresponding to the continuous model A,

$$\partial_t \varphi = \nabla^2 \varphi + a\varphi(1 - \varphi^2). \tag{31}$$

Starting from an uncorrelated Gaussian initial condition, one can solve the equation using a finite differences scheme and compute $p_l(t)$ for different block sizes. Using block scaling, we can determine θ_0 and θ. For both exponents, we find a value somewhat smaller than for the Glauber Ising model: $\theta_0 = 0.06 < 0.09$ and $\theta = 0.20 < 0.22$ (the value of θ has been also computed by Cornell [30]). For θ, the large time decay of $p_l(t)$ shows significant curvature and the effective exponent seems to increase with time. The scaling function cannot be superposed with the scaling function of the Ising model. Moreover, the fact that both models have different θ_0 shows that the *two-time correlations* of the global magnetization (which solely determine θ_0) are *different* in the scaling regime. This suggests that model A could be in a different universality class from the Ising model. This was also suggested by Rutenberg [31] in a recent paper, as he argued that *model-dependent* anisotropy in the correlation function (due for instance to the lattice) does not vanish in the scaling regime.

One-dimensional XY model

The one-dimensional nonconserved XY model is exactly soluble [1,32] and is quite special as for short-ranged initial correlation the structure factor does not exhibit conventional scaling, and the growth exponent $z = 4$ in contrast with general results for nonconserved vector spin systems.

The order parameter is a unitary two-dimensional vector φ, defined by its phase $\alpha(x, t)$. The equation of motion is simply a diffusion equation

$$\partial_t \alpha = \partial^2_{xx} \alpha, \tag{32}$$

with a Gaussian initial condition,

$$P(\{\alpha_k(0)\}) \propto \exp\left(-\sum_k \frac{\beta_k}{2} \alpha_k(0)\alpha_{-k}(0)\right). \tag{33}$$

The correlation function $C(r, t_1, t_2) = \langle \cos[\alpha(r, t_1) - \alpha(r, t_2)] \rangle$ depends on the initial condition. For a finite

correlation length in the initial condition $\beta_k = (\xi/2)k^2$, $C(r,t,t')$ does not have the usual scaling form [1]. As a consequence, the normalized correlator of the magnetization is not a function of t/t'.

Now, if the initial correlator has a power law decay $C(r,0) \sim r^{-\gamma/\pi}$, the general scaling form is recovered, with

$$C(r,t,t') = f\left(\frac{r}{\sqrt{t+t'}}\right)\left[\frac{4t_1t_2}{(t+t')^2}\right]^{\frac{\gamma}{4\pi}}, \qquad (34)$$

with $f(x) \propto x^{-\gamma/\pi}$ [1]. For $\gamma > \pi$, the spatial correlations of the order parameter decay sufficiently fast for the central limit theorem to be valid (see *e.g.* [33]). Therefore, the magnetization is still Gaussian, and the global persistence probability, now defined as the probability that a component of the magnetization has never changed sign, is determined by the normalized correlator

$$a_g(t,t') = \left[\frac{4tt'}{(t+t')^2}\right]^{\frac{\gamma-\pi}{4\pi}}, \qquad (35)$$

which continuously depends on γ. This is the *local* correlator of the diffusion equation in dimension $d = (\gamma-\pi)/\pi$. Since the diffusing field is Gaussian, we conclude that θ_0 for the one dimensional XY model with $C(r,t=0) \sim r^{-\gamma/\pi}$ is equal to θ for the diffusion equation in dimension $(\gamma-\pi)/\pi$, a quite surprising result.

Before moving to finite temperature, let us mention that in a recent paper [34], Hinrichsen *et al.* have used our method to study persistence for a directed percolation model.

5 Block persistence at finite temperature

Although block persistence is useful even at zero temperature, our main concern remains finite temperature, which we discuss now. The main idea is that because the correlation length is finite at finite time, the relative fluctuations of the block variables vanish as $l^{-d/2}$ when the size of the blocks is increased, and therefore the large block limit corresponds to no fluctuations at all, *i.e.*, zero temperature. In fact this picture is just a naive justification of the renormalization group flow for coarsening. One has to single out $T = T_c$ where the *relative* fluctuations diverge, as the equilibrium magnetization vanishes, and for which a different scaling arises.

5.1 $T < T_c$

Let us first consider $0 < T < T_c$. The difficulty in defining a persistence exponent comes from the fact that a spin may flip due to thermal fluctuations, leading to an exponential decay $p(t) \sim \exp(-t/\tau)$. Indeed, at $T = 0$, a spin flips only when it is crossed by an interface between a + and a − domain, whereas at finite temperature, the dominating process at late time, when the domains are large, is the flip of a spin within a domain due to thermal fluctuations. Therefore, at low temperature, it is natural from classical kinetics intuition to expect an Arrhenius law $\tau \sim \exp(-\Delta\mathcal{E}/T)$, where $\Delta\mathcal{E}$ is the energy barrier to flip a spin (or a block) within an ordered domain. As $T \to 0$, τ diverges and p crosses over to a power law.

Arrhenius laws are common enough in physics and chemistry, and arise each time a fluctuating process has to cross a finite barrier. It is useful, though, to work out the random process viewpoint, to clearly understand how τ should behave with l.

Let us consider a block of linear size l, and spin block variables φ_l. When $L(t)$ is large enough, the system can be considered locally at equilibrium inside a domain, and, since there are no long-range correlations, $\langle\varphi_l(t)\rangle \approx l^d\langle\varphi\rangle_{eq}$ and $(\Delta\varphi_l)^2 = \langle\varphi_l^2(t)\rangle - \langle\varphi_l(t)\rangle^2 \approx l^d(\Delta\varphi)^2$. Therefore the relative fluctuation of φ_l has the scaling $\Delta\varphi_l/\langle\varphi_l\rangle \propto \sqrt{T/l^d}$.

Thus $p_l(t)$ is essentially the probability that the stationary random process $\varphi_l(t)$ with mean value of order l^d and fluctuations of the same order has never crossed zero. In other words, it is the survival probability of a stationary walker $X(t) = (\varphi_l(t) - \langle\varphi_l\rangle)/\langle\varphi_l(t)\rangle$, with zero mean and a mean square fluctuation $\langle X^2\rangle = aT/l^d$, and an absorbing boundary at $x = 1$. To simplify, let us assume that $X(t)$ is Gaussian and Markovian. Then one can write a simple Langevin equation,

$$\dot{X}(t) = -\gamma X(t) + \eta(t) \qquad (36)$$

with a Gaussian white noise $\eta(t)$ with $\langle\eta(t)\eta(t')\rangle = 2aT/l^d\,\delta(t-t')$. In exponential time $u = e^{2\gamma t}$, the new random variable $Y(t) = 2\gamma\sqrt{u}X$ performs a simple random walk,

$$\dot{Y} = \xi(t), \qquad (37)$$

where $\xi(t) = \eta/\sqrt{u}$ is a new Gaussian white noise, with $\langle\xi(u)\xi(u')\rangle = 4\gamma aT/l^d\delta(u-u')$. Hence, $p_l(u)$ is the survival probability of a simple one-dimensional random walker with diffusion coefficient $D = 2aT/l^d$, starting from $x = 0$ with a moving absorbing wall at $x(u) = \sqrt{u}$. The survival probability is just,

$$S(u) = \int_{-\infty}^{\sqrt{u}} P(x,u)dx, \qquad (38)$$

where $P(x,u)$ is the presence probability of the walker. $P(x,u)$ is the solution of the diffusion equation with an absorbing boundary condition at $x = \sqrt{u}$. When the wall motion is much faster than the diffusion of the walker, *i.e.* $D \ll 1$, which corresponds to small T or large l (small fluctuations), $P(x,u)$ can be well approximated by a Gaussian distribution with a time dependent weight $S(u)$ [35],

$$P(x,u) = \frac{S(u)}{\sqrt{4\pi Du}}e^{-\frac{x^2}{4Du}}, \qquad (39)$$

where $S(u)$ is determined by equating the mass loss rate with the flux of mass through the moving wall.

At large u, $S(u)$ decays with a power law $u^{-\beta}$ and $\beta = (4\pi D)^{-1/2}\exp(-1/4D)$. Since $p_l(t) = S(e^{2\gamma t})$, we recover the heuristic Arrhenius law with,

$$\tau = 1/(2\gamma\beta) = \sqrt{\frac{2\pi a T}{\gamma^2 l^d}}\exp[l^d/(8aT)]. \qquad (40)$$

The constant a is a slowly varying function of the equilibrium correlation length but does not depend on l for large l.

The important point is that the effective temperature entering the Arrhenius law of the spin blocks is cut by a factor l^d and that τ diverges very quickly when l is increased, leading to a fast cross over to the $T = 0$ behavior. Admittedly, the actual stochastic process $\varphi_l(t)$ is certainly non-Markovian. However, for l much bigger than the equilibrium correlation length, it is nearly Gaussian from the central limit theorem. Moreover, its correlator $C(t) = \langle\varphi_l(t)\varphi_l(0)\rangle - \langle\varphi_l(0)\rangle^2$ can be bounded by two Markovian exponential correlators (because there is no long range correlation in time at equilibrium), and thus the Arrhenius law still holds with proper constants inserted (although the power law in the prefactor may be modified), from the discussion of Section 2.

For $t \ll \tau$, $p_l(t)$ is expected to behave in a similar way as for $T = 0$, and we expect

$$p_l(t) \sim l^{-z\theta_0} f(t/l^z)\exp[-t/\tau(l,T)], \qquad (41)$$

with two different cross-over times. However, in the scaling limit $l \to \infty$, τ diverges much faster than l^z, entailing that the exponential part does not scale. Hence, the scaling form of $p_l(t)$ should be equation (15). Moreover, from the universality of the domain wall dynamics for $T < T_c$, the scaling function g should be the same as at zero temperature, up to an overall temperature dependent multiplicative factor. As for the scaling function f, we have to take into account a temperature dependent multiplicative constant in $L(t)$ (see below).

To illustrate these ideas, we have performed simulations of the *two-dimensional Glauber Ising model* at finite temperature on a 1000^2 lattice. Figure 7 presents results at $T = 2T_c/3$ for blocks of size l =1, 3, 5, 7, 9, 11 and 13. The exponential decay is clearly visible for $l = 1$ and $l = 3$. However, for larger blocks, τ is bigger than the simulation time, and $p_l(t)$ has the $T = 0$ behavior, with a power law decay with exponent θ fully compatible with the $T = 0$ value ($\theta = 0.22$), for $t \gg l^2$, and a power law decay with exponent θ_0, for $t < l^2$, just as expected. Figure 8 shows the scaling function at $T = T_c/2$ (where the approach to scaling is faster) obtained with the zero temperature value $\theta_0 = 0.09$,for l=7, 9, 11 and 13. The data collapse is really excellent.

The temperature universality of the scaling function is illustrated in Figure 9. We plot the quantity $f = l^{2\theta_0}p_l(t)$ *versus* $x = t/l^2$ for a set of zero temperature data, and $f = a_1 l^{2\theta_0}p_l(t)$ *versus* $x = a_2 t/l^2$ for $T_c/2$ data, for blocks of size 7, 9, 11 and 13. The constants a_1 and a_2 are the same for all sizes, and are adjusted to superpose the two sets of data. a_2 arises from the temperature dependence

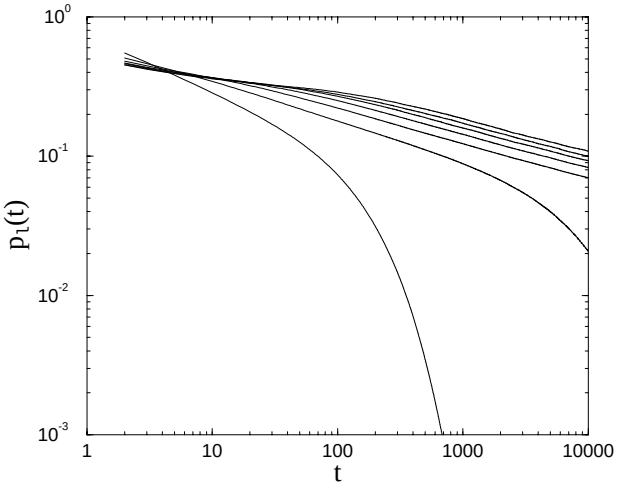

Fig. 7. $p_l(t)$ for the two dimensional Ising model at $T = 2T_c/3$, and block sizes l =1, 3, 5, 7, 9, 11 and 13. The exponential decay of $p_l(t) \sim e^{-t/\tau}$ is clearly visible for $l = 1$ and $l = 3$, however for $l = 5$, the exponential regime is already repelled at times longer than the simulation time, in agreement with the expected fast divergence $\tau \sim \exp(al^2/T)$. For $l > 5$, only the power law zero-temperature like regime is to be seen.

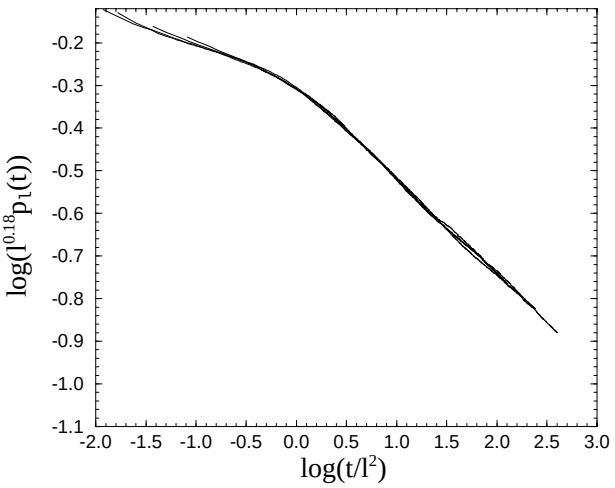

Fig. 8. $p_l(t)$ expressed in scaling form for $T = T_c/2$, and block sizes $l = 7, 9, 11, 13$, using the same value for $2\theta_0 = 0.18$ as in the $T = 0$ case. Note the similarity with the $T = 0$ scaling function of Figure 1.

of the prefactor of $t^{1/2}$ in $L(t)$ (the natural time variable) and a_1 is the overall temperature dependent multiplicative discussed above. The superposition obtained is really excellent and assesses the expected universality of the scaling of block persistence, in a very similar way as what is known for the equal-time two-point spin correlation function [1].

Thus, block scaling leads to a definition of θ at finite temperature as the exponent of the algebraic decay of the scaling function $f(x)$. For the two-dimensional Ising model, the temperature independence of θ obtained with this method confirms the results obtained with Derrida's definition, but the universality is stronger, since

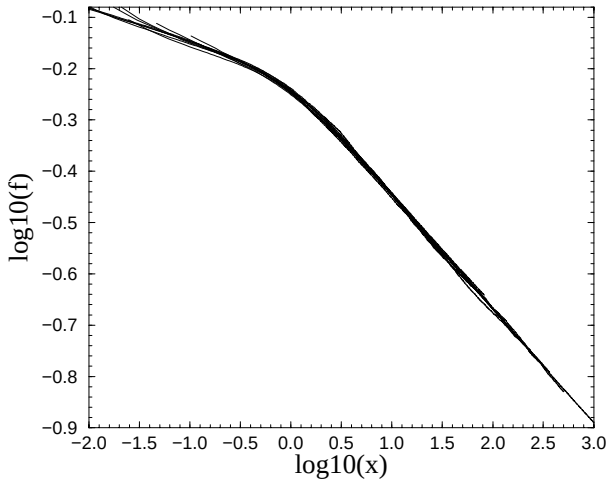

Fig. 9. Universality of the scaling function for block persistence. We show the superposition of scaling data corresponding to $T = 0$ and $T = T_c/2$. For $T = 0$, we (double-log) plot $f = l^{2\theta_0} p_l(t)$ *versus* $x = t/l^2$, whereas for $T = T_c/2$ we plot $f = a_1 l^{2\theta_0} p_l(t)$ *versus* $x = a_2\, t/l^2$, with $a_1 = 1.07$ and $a_2 = 1.26$. The excellent superposition of the two scaling functions assesses the universality.

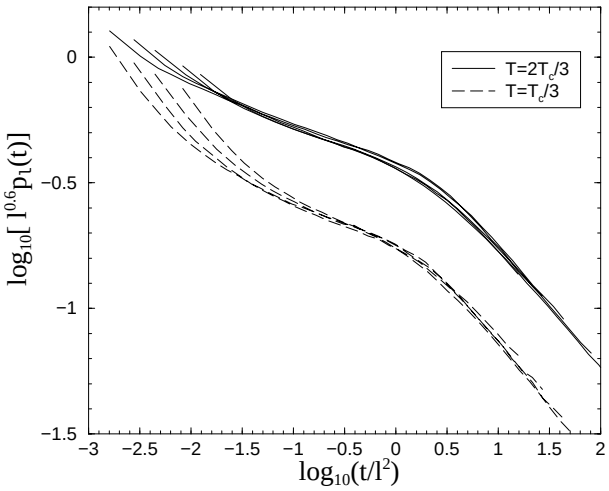

Fig. 10. Scaling of block persistence for the 7-state Potts model at finite temperature $T = T_c/3$ and $T = 2T_c/3$, from simulations on a 1000^2 lattice for blocks of size l =9, 11, 15, 19 and 25 (13 samples). Data are noisier than for the Ising model, but acceptable scaling is obtained with $\theta_0 = 0.3$.

the *whole block persistence scaling function* is universal. This universality arises from general arguments and should be observed for generic systems.

Note that universality would rather be expressed in terms of $L(t)$ than in terms of t. This is especially relevant for the *three dimensional Glauber Ising model* with nearest neighbors interactions on the cubic lattice. Numerical simulations [3,14] at $T = 0$ lead to $\theta \approx 0.17$, whereas at finite temperature our method leads to $\theta_{T>0} \approx 0.26$ in agreement with results obtained by Stauffer [10] using Derrida's definition. For this problem, seemingly due to lattice effects, $L(t)$ does not grow as $t^{1/2}$ at zero temperature, but as $t^{0.33}$ [36]. At finite temperature, lattice effects are overcome, and one recovers the usual growth law. Now if the block scaling function g is universal, one should have the same value of θz at any temperature. From numerical results we obtain $\theta z = 3.0 \times 0.17 = 0.51$ at $T = 0$ and $\theta z = 0.26 \times 2 = 0.52$ at finite T, which actually confirms this *universality*. Note that these values of θz are in good agreement with an approximate continuous theory [14].

Block persistence is also very useful to study persistence for the *q-state Potts model*, as zero temperature dynamics show blocking effects at zero temperature on the square lattice with nearest neighbor interactions [37]. Working at finite temperature is a more satisfactory way of overcoming blocking effects than changing the lattice type or including next nearest neighbors interactions. Derrida used his comparison method to study the $q = 7$ Potts model. His data seemed to suggest a temperature dependence of θ [9].

On the basis of the present work, we would rather expect θ to be independent of T, at least with our definition. To address this question, we have performed simulations of the $q = 7$ Potts model at $T = T_c/3$ and $T = 2T_c/3$, on a 1000^2 lattice. We have computed $p_l(t)$ for l =1, 3, 9, 11, 15, 19 and 25, where the block variables are defined through a majority rule. For both temperature, acceptable scaling is obtained for $\theta_0 = 0.3$, but the scaling is not as good as for the Ising model, and would surely be improved by using larger block and simulating longer times. Moreover, nonscaling transients extend over quite a long period of time for $T = 2T_c/3$. The extraction of θ from the decay of $p_9(t)$ leads to $\theta(2T_c/3) \approx 0.485$ and $\theta(T_c/3) = 0.47$. The discrepancy is not really significant compared to numerical uncertainties, and is much smaller anyway than for Derrida's data [9], who found $\theta(2T_c/3) \approx 0.55$ and $\theta(T_c/3) \approx 0.4$. Hence, θ does not seem to depend on temperature. This is confirmed by the comparison of the two scaling functions, which can once again be superposed through a global rescaling.

Finally, the value of θ compares well with zero temperature data obtained by Derrida *et al.* [37]. These authors simulated the next nearest neighbor interactions Potts model to avoid blocking effects. Data for $p(t)$ showed significant curvature, due to the fact that the effective z exponent increases with time, and better results were obtained for the exponent φ defined as $p(t) \sim L(t)^{-\varphi} \sim E(t)^{\varphi}$, where E is the energy difference with the fundamental state. These authors found $\varphi = 1.01$ for the $q = 7$ Potts model. Assuming $\varphi = \theta z$ with the asymptotic value $z = 2$, their data lead to $\theta \approx 0.5$, in acceptable agreement with our results at finite temperature.

5.2 $T = T_c$

The naive kinetic argument giving the scaling of the relaxation rate of $p_l(t)$ is bound to break down at the critical temperature for several reasons. We know from explicit renormalization group analysis [38] that T_c is a fixed point

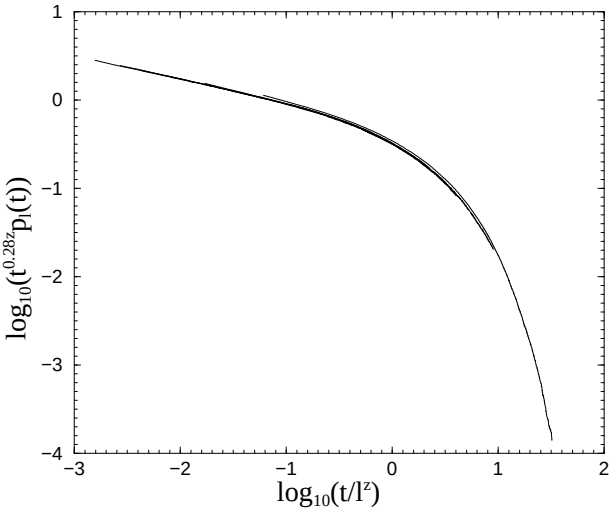

Fig. 11. Scaling of the block persistence probability at T_c from simulations of the two-dimensional Ising model on a 1000^2 lattice (15 samples) and blocks of size l =5, 9, 13, 17, 21 and 27. We have taken $z = z_c = 2.17$ and adjusted θ_c. The best collapse is obtained for $\theta_c = 0.28$.

for the dynamics. Hence in contrast with $T < T_c$, the thermal decay of the persistence probability must scale. Since the equilibrium magnetization is zero, one can no longer make a distinction between a slow flip mode due to interface motion and a fast flip mode due to thermal equilibrium fluctuations within domains. There are no domain walls in the system, and the relevant length scale is the time dependent correlation length $\xi(t) \sim t^{1/z_c}$. During the dynamics, patches of correlated spins of length $\xi(t)$ appear in the system. These large patches have a large life time due to *critical slowing down*. If we consider a block of spins inside one of these patches, the typical time required to flip the spin of the block should be roughly speaking of the order of the time required to relax a fluctuation of wave vector $\approx 2\pi/l$ in the critical equilibrium state. Hence, one should have $p_l(t) \sim \exp(-\omega_c(2\pi/l)t)$ where $\omega_c(k)$ is the characteristic critical relaxation frequency and scales as k^{z_c} [38]. This leads to an exponential decay $p_l(t) \sim \exp(-at/l^{z_c})$. Therefore, as predicted by the renormalization group argument, the exponential decay of the persistence probability scales and we must have

$$p_l(t) \sim l^{-\theta_c z_c} g(\xi(t)/l) = l^{-\theta_c z_c} f(t/l^{z_c}) \qquad (42)$$

in the scaling limit $l \to \infty$ with a fixed ratio ξ/l, where $f(x) \sim x^{-\theta_c}$ when $x \to 0$ and $f(x) \sim e^{-ax}$ when $x \to \infty$. The exponent θ_c is the global persistence exponent at T_c [7].

This scaling theory was checked for the two-dimensional Glauber Ising model from simulations on a 1000^2 lattice and blocks of size 1, 5, 9, 13, 17, 21 and 27. Fifteen samples were averaged to obtain the data shown in Figure 11. Excellent scaling is found with $z_c = 2.17$ and $\theta_c = 0.28$, which is also in agreement with the value of θ_c obtained by fitting the small x power law. The scaling function has an exponentially fast decay at large argument, as expected. The value obtained for θ_c yields $\theta_c z_c \approx 0.607$, some 20% bigger than the value found by direct determination of the global persistence probability [7,18,19], and the reason for this discrepancy is unclear.

6 Conserved models

The amount of certitudes we have for conserved order parameter dynamics is by far much smaller than for the nonconserved case. The large n limit is quite pathological as it exhibits multiscaling, which is not observed in simulations. The only exact result is the celebrated Lifshitz-Slyozov-Wagner (LSW) theory [39,40] for the limit of a vanishing concentration of minority phase. In this limit, well separated droplets of minority phase are embedded in a matrix of majority phase. This spatial structure is very different from the labyrinth-like domain structure of the equal concentration case. The typical length scale $L(t)$ grows as $t^{1/3}$ ($z = 3$) and one can compute the scaling function for equal time correlations. Recently, Lee and Rutenberg [41], have shown that $\lambda = d$ for LSW.

For finite concentrations of the minority phase, and especially for the zero-magnetization case, the situation was more controversial. While the $t^{1/3}$ growth law seems well established since the numerical work of Huse [42], no conclusive result for λ is available. Numerical simulations of conserved models are difficult because the dynamics are much slower than for the nonconserved models, and that corrections to scaling are important even at long simulation times (see below). Moreover, the spin-exchange Kawasaki dynamics freeze at zero temperature, and simulations must be performed at finite temperature, and the standard definition of persistence cannot be used. This explains why results for conserved persistence reduce to an analytical computation of θ in the LSW theory [41]. Actually, in the absence of numerical simulations, the question of whether the persistence probability has a power law decay or not is open, even if the answer is intuitively yes. Using block persistence, one can now extract information from finite temperature simulations, and study persistence for the Kawasaki $d = 2$ model. Numerical limitations due to slow dynamics remain, however.

6.1 Block scaling for conserved models

Naively, we expect the same scaling as for nonconserved dynamics (Eq. (15)), with the only difference that θ_0 is equal to zero due to the conservation of the global magnetization. It must be pointed out, however, that this assumption is incorrect generally speaking, and the proper scaling is

$$p_l(t) \sim l^{-z\theta'} f[t/l^z], \qquad (43)$$

where θ' is an *a priori nontrivial* exponent not directly related to global persistence. The scaling function $f(x)$

diverges as $x^{-\theta'}$ when $x \to 0$, and is expected to decay as a power law $x^{-\theta}$ at large x, θ being the local persistence exponent.

The occurrence of a nontrivial θ' arises convincingly from a simple analytical argument based on the scaling of correlations. From the scaling of the Fourier transform of the two points correlator, $C(\mathbf{k}, t, t') = L^d f(kL, L'/L)$, we obtain for the correlator of block variables

$$C_l(t,t') = \int_{kl<1} \frac{d^d\mathbf{k}}{(2\pi)^d} C(\mathbf{k},t,t') = \int_{u<(l/L)} \frac{d^d\mathbf{u}}{(2\pi)^d} f(u, L'/L). \quad (44)$$

Then, in the limit $l \to \infty$ with a fixed ratio $x = L/l$ and $y = L'/l$, we obtain the expression of the correlator of block variables in the block scaling regime,

$$C_l(t,t') \propto \int_0^{1/x} u^{d-1} f(u, x/y) du. \quad (45)$$

This expression suggests a scaling form $l^{-\alpha} g(L/l)$ for $p_l(t)$. Moreover, we can investigate the asymptotic small x behavior of the scaling function $g(x)$, since in this limit the block variables are Gaussian and the persistence is fully determined by $C_l(t,t')$. Assuming quite generically that $f(u, x/y) = u^j \kappa(x/y) + o(u^j)$, the small x and y asymptotic behavior of the normalized correlator is

$$a_l(x,y) \sim \left(\frac{y}{x}\right)^{(j+d)/2} \kappa(x/y). \quad (46)$$

Since $a_l(x,y)$ is a function of x/y, $p_l(t)$ scales as $c_0 x^{-\alpha} \sim t^{-\theta'}$, where α is *a priori nontrivial* and $\theta' = \alpha/z$, leading to the scaling of equation (43).

Equation (46) holds for nonconserved dynamics as well as for conserved dynamics. For nonconserved coarsening dynamics, the variance of the global magnetization $C(\mathbf{k} = 0, t, t)$ usually grows as $L(t)^d$, leading to $j = 0$. It is easily seen that the correlator of equation (46) is precisely equal to the normalized correlator of the global magnetization. Therefore, θ' is equal to the global persistence exponent θ_0, as argued before.

For conserved dynamics, however, the normalized correlator of the magnetization is 1, whereas $a_l(x,y)$ is generally speaking a nontrivial function of x/y. Consequently, θ' is nontrivial and not equal to $\theta_0 = 0$. The interpretation of this result is that, for a block of finite size, three regimes are observed. At early times, the magnetization remains constant as if the block were infinite. In a second regime, the finite size of the block becomes relevant, significantly breaking the magnetization conservation, and the persistence decays as $t^{-\theta'}$. Finally, for $L \gg l$, the local persistence decay is recovered. The point we make is that the first constant magnetization stage does not generally scale, and only the second "pseudo-global" persistence behavior appears in the scaling function. Nevertheless, for a class of systems, $\kappa(x/y)$ may be equal to $(x/y)^{(d+j)/2}$, leading to $\theta' = 0$. We present now some illustrative numerical simulations for conserved dynamics.

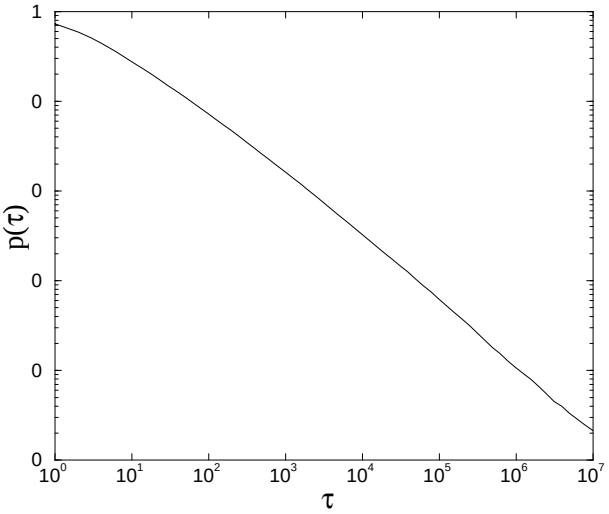

Fig. 12. Numerical results for the fraction of persistent spins in the domain model corresponding to the $T \to 0$ limit of the Kawasaki Ising chain (10^6 sites, 12 samples). We find a power law decay at large time with $\theta = 0.73$.

6.2 1D-Kawasaki dynamics

One-dimensional spin-exchange dynamics (Kawasaki dynamics) are peculiar, as they do not coarsen at any temperature. Indeed, since the critical temperature is zero, coarsening does not occur at finite temperature, whereas the system freezes at zero temperature. However, Majumdar *et al.* [43,25] have shown that coarsening occurs in the $T \to 0$ limit in the rescaled time $\tau = t\exp(-4J/k_bT)$. The obtained dynamics is equivalent to a domain diffusion model of Cornell *et al.* [44]. In this model, domains of length L perform random walks with a diffusion constant proportional to $1/L$ and coalesce. At small finite temperature, this corresponds to the fact that a domain of + phase moves through the diffusion of an isolated − spin detached with probability $\exp(-4J/k_bT)$ from a neighboring − domain and reaching the other neighboring domain after about L^2 steps of a random walk [43,44]. Majumdar *et al.* have argued that for this model $\lambda = d$, which they have checked numerically [43].

The local persistence exponent of the one-dimensional Kawasaki dynamics can be defined through this domain model. We present on Figure 12 results of simulations of the model on a $L = 10^6$ chain (10 samples). We observe a power law decay with $p(\tau) \propto \tau^{-\theta}$, with $\theta = 0.73$. To our knowledge, this is the first numerical demonstration of the existence of a persistence exponent for a conserved model, confirming the result obtained in the LSW limit by Lee and Rutenberg [41]. Note that the persistence exponent is much bigger than for Glauber dynamics ($\theta = 3/8$) (see below). The complexity of the aggregation model leaves little hope of obtaining the exact value of θ for this model.

We have computed the block persistence probability $p_l(t)$ for l =101, 131, 161, 191 and 211. Results are presented in scaling form in Figure 13. For increasing l, the approach to scaling is very slow and although we have

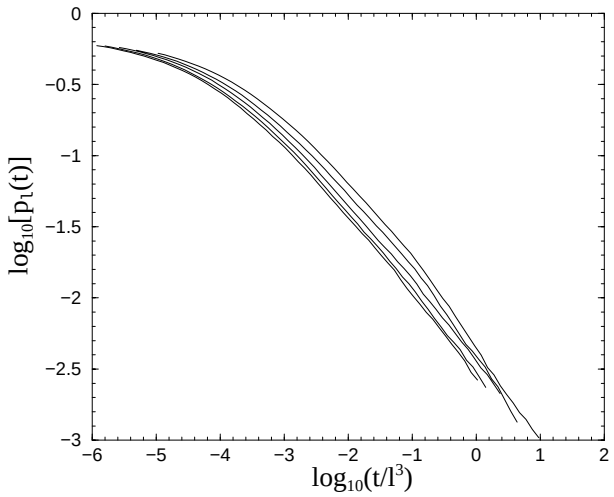

Fig. 13. Block persistence probability in scaling form (with $\theta' = 0$) for the domain model corresponding to the $T \to 0$ limit of the Kawasaki Ising chain. Simulations were carried out for a 10^6 sites chain (12 samples) for blocks of size l =101, 131, 161, 191 and 211 (from top to bottom). The approach to scaling is very slow when l is increased, but it is clearly visible that the scaling function goes to a constant ($\theta' = 0$) as expected since $\lambda = d$.

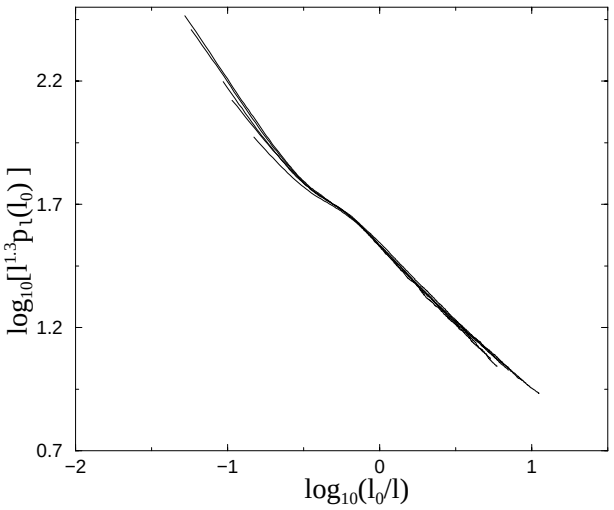

Fig. 14. Scaling of the block persistence probability for the domain aggregation model equivalent to the 1D zero temperature Cahn-Hilliard dynamics. Simulations were carried out on a 10^6 sites chain, for blocks of size 101, 131, 161, 191, and 211. For this model $\lambda < d$ and the scaling function for block persistence has a power law divergence at small x. Good scaling is obtained with $\theta' = 1.3$, and we find $\theta = 0.62$.

used very large blocks, the data collapse is poor. The presented scaling was obtained for $\theta' = 0$, and the collapse was worse for positive θ'. A zero value of θ' is also suggested by the fact that the scaling function $f(x)$ appears to tend to a constant when $x \to 0$.

6.3 Deterministic domain model

To clearly illustrate the possible occurrence of a nontrivial θ', we have also simulated the zero temperature one-dimensional Cahn-Hilliard equation (model B). Majumdar and Huse [25] have shown that the dynamics could be mapped on a deterministic domain aggregation model. In one step of the dynamics, the shortest domain I_0 of length l_0 is localized and removed, the left (length l_l) and right (length l_r) neighbors are merged. The length l_0 is dispatched between the right (l_{rr}) and left (l_{ll}) second neighbors (which have the same sign as I_0), according to $l_{ll} = l_{ll} + l_{0l}$ and $l_{rr} = l_{rr} + l_{0r}$, with $l_{0r} + l_{0l} = l_0$ and $l_{0r} : l_{0l} = l_l : l_r$. These domain dynamics reflect the fact that the shortest domain shrinks due to diffusion fluxes from I_0 to its second neighbors through its first neighbors. The fluxes are proportional to e^{-2l_0}, which makes the shortest domain shrink much faster than other domains. The flux to the right (resp. left) is proportional to $1/l_l$ (resp. $1/l_r$), which leads to the above ratio of l_{0r} and l_{0l}.

We have performed simulations of the domain model on a chain of 10^6 sites (20 samples), for blocks of size 101, 131, 161, 191 and 211. The scaling function presented in Figure 14 is qualitatively very different from the one in Figure 12. In agreement with the general discussion above and equation (43), we find a cross-over between two exponents θ' and θ. The best data collapse is obtained for $\theta' = 1.3$, whereas we find $\theta = 0.62$ (remark however that the large x decay shows curvature). Unlike one-dimensional Kawasaki dynamics, this system clearly exhibits a nonzero θ', and the scaling is really convincing. Remark that one-dimensional Kawasaki and Cahn-Hilliard dynamics are also known to belong to different classes as far as the autocorrelation exponent λ is concerned: Kawasaki dynamics have a trivial $\lambda = d$ whereas λ is nontrivial (≈ 0.67) for Cahn-Hilliard dynamics [25].

6.4 Two-dimensional Kawasaki dynamics

Effective zero-temperature domain models cannot be used successfully to avoid the freezing of the two-dimensional Kawasaki dynamics, because of the complicated geometry of the domains. Using block persistence simulations can be performed at finite temperature. As mentioned above, simulations are difficult because the Kawasaki dynamics are very slow, and do not reach the pure $t^{1/3}$ regime. Therefore, it is difficult to observe block scaling and to extract the persistence exponent, and we have to be satisfied with qualitative results. Figure 15 presents data obtained for a 1000^2 systems with a simulation time of 500 000 Monte-Carlo steps for blocks of size 3, 5, 7, 9, 11, and 15, and 2000 steps for l =15, 21, 25, 35, 45 and 55. The cross-over in the behavior of $p_l(t)$ corresponding to $L(t) \sim l$ is visible for small blocks. At large time, we observe a power law decay with a persistence exponent $\theta \approx 0.5$. The actual value of θ is certainly bigger since the effective z exponent increases with time and is still far from its asymptotic value $z = 3$ ($1/z \approx 0.25$ at the end

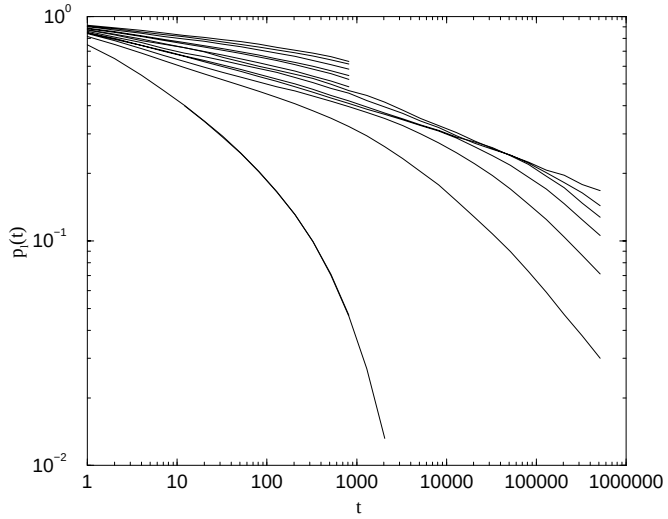

Fig. 15. Results of simulations of the Kawasaki spin-exchange dynamics on a 700^2 lattice, for blocks of size l =1, 3, 5, 7, 9, 11 and 13 (one sample, 500 000 steps), 15, 21, 25, 35, 45, 55 (12 samples, 2000 steps).

of the simulation). Still, we acknowledge that these data are not very conclusive.

It is not surprising to find a power law decay of the persistence probability (in the block scaling or $T \to 0$) limit, because of the slow motion of interfaces, as for the nonconserved case. What is less intuitive, is that θ is much bigger for conserved dynamics than for nonconserved dynamics. However, one can understand that fast dynamics may lead to a small θ, if one realizes that a fast moving domain wall will be ineffective in decreasing $p(t)$ if it wipes several times regions of spins that have already flipped. Once again, we see that θ reflects very subtle effects.

At early times, when l is increased, we do not seem to have a power law regime, suggesting $\theta' = 0$ for two-dimensional Kawasaki dynamics, but because $L(t)$ has strong corrections to scaling, it is not clear that correlations correctly scale in the time regime observe, and one has to be careful. For large blocks and large time with $L(t) \sim t^{1/3} \ll l$, one might observe a power law.

7 Conclusion

In this article we have introduced the notion of block persistence as a generalization of global and local persistence probabilities, and as a way of giving a meaning to the persistence exponent θ *at finite temperature*. Theoretical arguments as well as results of simulations suggest that the persistence exponents and the whole scaling function of block persistence are temperature independent in the whole $T < T_c$ phase, which is conceptually speaking very satisfactory. We have also shown that persistence exponents arise for conserved models as well. It would also be interesting to use block persistence for systems with continuous symmetries, for instance the Heisenberg model, or more generally the $O(n)$ model. A block spin would be considered to flip whenever one of its components changes sign. Actually, this definition was used in this article for the large-n model.

Finally, the important question may be: What do we learn from persistence? In fact, the justification for studying such crude models as the one used in coarsening is universality, which states that most of the fine details of the system are irrelevant for the study of the scaling regime. Then the stake is to identify universality classes, to understand the parameters that determine them, and also to identify universal quantities. The theoretical and numerical study of persistence shows us, because it probes temporal correlations very sensitively, that model universality is not as wide as it may have been hoped *a priori* from equilibrium-based intuition. While universality with respect to initial conditions, or interactions range, seems to hold in most cases, the present work suggests that the continuous model A and the Glauber Ising model in two dimensions are in different universality classes (at least at zero temperature), even if they have the same dimensionality, the same conservation law and both short-ranged interactions. Hence, the existence of the lattice seems to affect correlations even at large time. This could be related to the pertaining of anisotropy at large time claimed by Rutenberg in a recent work [31].

The authors have benefitted from interesting discussions with S. Majumdar, B. Derrida, S. Cornell.

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